

Quasinormal modes around the BTZ black hole at the tricritical generalized massive gravity

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Abstract

Employing the operator method, we obtain log-square quasinormal modes and frequencies of a graviton around the BTZ black hole at the tricritical point of the generalized massive gravity. The log-square quasinormal frequencies are also obtained by considering a finite temperature conformal field theory. This shows the AdS/LCFT correspondence at the tricritical point approximately. We discuss a truncation process to the unitary theory on the BTZ black hole background.

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1 Introduction

Critical gravities have been the subject of active interest because they were considered as toy models for quantum gravity [1, 2, 3, 4]. According to the AdS/LCFT correspondence, one finds that a rank-2 logarithmic conformal field theory (LCFT) is dual to a critical gravity [5, 6, 7]. However, one has to deal with the non-unitarity issue of these theories because the LCFT is in general non-unitary.

Recently, a higher-derivative critical (polycritical) gravity was introduced to provide multiple critical points [8] which might be described by a higher-rank LCFT. The rank of the LCFT refers to the dimensionality of the Jordan cell. The LCFT dual to critical gravity has rank-2 and thus, an operator has one logarithmic partner. The LCFT dual to tricritical gravity has rank-3 and thus, an operator has two logarithmic partners. An odd-rank LCFT allows for a truncation to a unitary conformal field theory (CFT) [9]. A six-derivative gravity in three dimensions was treated as dual to a rank-3 LCFT [10], while four-derivative critical gravity in four dimensions was considered as dual to a rank-3 LCFT [11]. Furthermore, it is shown that a consistent unitary truncation of polycritical gravity was possible to occur at the linearized level for odd rank [12]. On the other hand, a non-linear critical gravity of rank-3 in three and four dimensions was investigated in [13], showing that truncation which appears to be unitary at the linear level might be inconsistent at the non-linear level. It is worth noting that a tricritical gravity was first mentioned in the six-derivative gravity in six dimensions [14].

It is important to note that one can construct a rank-3 parity odd theory in the simple context of four-derivative gravity as generalized massive gravity (GMG) in the three dimensional anti-de Sitter (AdS_3) spacetimes [15]. In fact, the GMG is a combination of topologically massive gravity (TMG) [16] and new massive gravity (NMG) [17]. There exists one parity-odd tricritical point in the GMG parameter space where the theory propagates one left-moving massless graviton as well as right-moving massless graviton, and two logarithmic modes associated with left-movers known as \log and \log^2 boundary behaviors on the AdS_3 background. Its dual theory is a rank-3 LCFT [18]. A truncation of tricritical GMG is made by imposing $Q_L = 0$ with Q_L the Abbott-Deser-Tekin charge [10]. After truncation, one has found the left-moving sector with a CFT which is unitary, in addition to the right-moving massless sector which dictates the chiral gravity.

Before we proceed, we would like to mention the following difference in AdS/LCFT correspondence:

- tricritical gravity on the $\text{AdS}_3 \rightarrow$ a rank-3 zero temperature LCFT
- tricritical gravity on the BTZ black hole \rightarrow a rank-3 finite temperature LCFT.

In this work, to confirm the AdS/LCFT correspondence [19], we obtain \log^2 -quasinormal modes and frequencies of a graviton around the BTZ black hole (instead of the AdS_3) at the tricritical GMG by employing the operator method. In order to obtain the quasinormal modes precisely, one needs to know a rank-3 finite temperature LCFT while a rank-2 finite temperature LCFT was known in Ref. [20]. Here, we show that the \log^2 -quasinormal frequencies are also obtained by considering a finite temperature CFT [21].

2 Generalized massive gravity

We consider the generalized massive gravity (GMG) action

$$S_{\text{GMG}} = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left[\sigma R - 2\lambda + \frac{1}{m^2} K + \frac{1}{\mu} L_{\text{CS}} \right], \quad (1)$$

where K (L_{CS}) is the new massive gravity (NMG) term (the Chern-Simons term) given by

$$K = R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2, \quad (2)$$

$$L_{\text{CS}} = \frac{1}{2} \epsilon^{\mu\nu\rho} \Gamma_{\mu\beta}^{\alpha} \left[\partial_{\nu} \Gamma_{\alpha\rho}^{\beta} + \frac{2}{3} \Gamma_{\nu\gamma}^{\beta} \Gamma_{\rho\alpha}^{\gamma} \right]. \quad (3)$$

Here m and μ are the two mass parameters, while σ is a dimensionless sign parameter which takes +1 for our purpose. We also use the convention of $\epsilon^{\rho\omega} = 1/\sqrt{-g}$ [22]. Replacing m^2 by $-m^2$ leads to the action in [15]. Also, λ is the cosmological constant. Its equation of motion takes the form

$$G_{\mu\nu} + \lambda g_{\mu\nu} + \frac{1}{2m^2} K_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = 0, \quad (4)$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \quad (5)$$

$$\begin{aligned} K_{\mu\nu} = & -\frac{1}{2} \nabla^2 R g_{\mu\nu} - \frac{1}{2} \nabla_{\mu} \nabla_{\nu} R + 2 \nabla^2 R_{\mu\nu} \\ & + 4 R_{\mu\alpha\nu\beta} R^{\alpha\beta} - \frac{3}{2} R R_{\mu\nu} - R_{\alpha\beta} R^{\alpha\beta} g_{\mu\nu} + \frac{3}{8} R^2 g_{\mu\nu}, \end{aligned} \quad (6)$$

and the Cotton tensor is given by

$$C_{\mu\nu} = \epsilon_{\mu}^{\alpha\beta} \nabla_{\alpha} \left(R_{\beta\nu} - \frac{1}{4} g_{\beta\nu} R \right). \quad (7)$$

In this work, we consider the BTZ black hole in the Schwarzschild coordinates

$$ds_{\text{BTZ}}^2 = - \left(-\mathcal{M} + \frac{r^2}{\ell^2} \right) dt^2 + \frac{dr^2}{-\mathcal{M} + \frac{r^2}{\ell^2}} + r^2 d\phi^2 \quad (8)$$

whose horizon is located at $r_+ = \ell\sqrt{\mathcal{M}}$. The $\mathcal{M} = -1$ case corresponds to the AdS_3 spacetimes, while $\mathcal{M} = 1$ provides a unity mass of the BTZ black hole. In these cases, one finds a relation among m^2 , λ , and Λ as

$$m^2 = \frac{\Lambda^2}{4(\lambda - \Lambda)}, \quad \Lambda = -1 \quad (9)$$

with $\ell = 1$. Also, the left-temperature, and right-temperature, and Hawking temperature are the same as

$$T_L = T_R = T_H = \frac{1}{2\pi}. \quad (10)$$

The line element (8) is expressed in terms of global coordinates

$$ds^2 = -\sinh^2(\rho)d\tau^2 + \cosh^2(\rho)d\phi^2 + d\rho^2, \quad (11)$$

where we have introduced the radial coordinate $r = \cosh \rho$ such that the event horizon of $r = r_+ = 1$ is located at $\rho = 0$, while the infinity is at $\rho = \infty$. Introducing the light cone coordinates $u/v = \tau \pm \phi$, the line element becomes

$$\begin{aligned} ds^2 &= \bar{g}_{\mu\nu} dx^\mu dx^\nu \\ &= \frac{1}{4}du^2 - \frac{1}{2}\cosh(2\rho)dudv + \frac{1}{4}dv^2 + d\rho^2. \end{aligned} \quad (12)$$

Then, the metric tensor (12) admits the Killing vector fields L_k , $k = 0, -1, 1$ for local $\text{SL}(2, R)_L \times \text{SL}(2, R)_R$ algebra as

$$L_0 = -\partial_u, \quad L_{-1/1} = e^{\mp u} \left[-\frac{\cosh(2\rho)}{\sinh(2\rho)}\partial_u - \frac{1}{\sinh(2\rho)}\partial_v \mp \frac{1}{2}\partial_\rho \right], \quad (13)$$

and $\bar{L}_0, \bar{L}_{-1/1}$ are similarly obtained by interchanging u and v ($u \leftrightarrow v$). Locally, they form a basis of the Lie algebra $\text{SL}(2, R)$ as

$$[L_0, L_{\pm 1}] = \mp L_{\pm 1}, \quad [L_1, L_{-1}] = 2L_0, \quad (14)$$

which will be used to generate the whole tower of quasinormal modes in three dimensions.

Now, we are going to expand $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ around the BTZ background in Eq. (12) and choose the transverse-traceless (TT) gauge

$$\bar{\nabla}_\mu h^{\mu\nu} = 0, \quad h^\mu{}_\mu = 0. \quad (15)$$

Here we wish to mention that the TT gauge is allowed, thanks to $\delta R(h) = 0$ which is obtained by tracing both sides of the linearized Einstein equation. Under the TT gauge, the linearized Einstein equation becomes the fourth-order differential equation

$$(\bar{\nabla}^2 - 2\Lambda) \left[\bar{\nabla}^2 h_{\mu\nu} + \frac{m^2}{\mu} \epsilon_\mu{}^{\alpha\beta} \bar{\nabla}_\alpha h_{\beta\nu} + \left(m^2 - \frac{5}{2}\Lambda \right) h_{\mu\nu} \right] = 0. \quad (16)$$

Introducing four mutually commuting operators of

$$\begin{aligned}(D^{L/R})_\mu^\beta &= \delta_\mu^\beta \pm \epsilon_\mu^{\alpha\beta} \bar{\nabla}_\alpha, \\ (D^{m_i})_\mu^\beta &= \delta_\mu^\beta + \frac{1}{m_i} \epsilon_\mu^{\alpha\beta} \bar{\nabla}_\alpha, \quad (i = 1, 2),\end{aligned}\tag{17}$$

the linearized equation of motion (16) can be written to be compactly

$$\left(D^R D^L D^{m_1} D^{m_2} h\right)_{\mu\nu} = 0.\tag{18}$$

Here, the mass parameters are given by

$$\begin{aligned}m_1 &= \frac{m^2}{2\mu} + \sqrt{\frac{m^4}{4\mu^2} - m^2 + \frac{1}{2}}, \\ m_2 &= \frac{m^2}{2\mu} - \sqrt{\frac{m^4}{4\mu^2} - m^2 + \frac{1}{2}}.\end{aligned}\tag{19}$$

The parameter space is shown in [10]. The two critical lines appear when $m_1 = m_2$. The NMG and TMG limits of the GMG are on the $\frac{1}{m^2}(x)$ -axis and $\frac{1}{\mu}(y)$ -axis, respectively. When a critical line intersects with one of them, either critical TMG or critical NMG is recovered. We are interested in two tricritical points

$$\text{point } 1 : m^2 = 2\mu = \frac{3}{2},\tag{20}$$

$$\text{point } 2 : m^2 = -2\mu = \frac{3}{2}.\tag{21}$$

At the point 1 of $m_1 = m_2 = 1$, \mathcal{D}^{m_1} and \mathcal{D}^{m_2} degenerate with \mathcal{D}^L , while at the point 2 of $m_1 = m_2 = -1$, \mathcal{D}^{m_1} and \mathcal{D}^{m_2} degenerate with \mathcal{D}^R . The presence of two tricritical points is a main feature of the GMG, but it is not a feature of the TMG or NMG. This implies that there is no tricritical point in the context of the TMG or NMG. Hereafter, we will focus on the tricritical point 1 because results for the point 2 could be obtained by exchanging L and R .

3 Log-square quasinormal modes at the tricritical GMG

We start with a first-order massive equation

$$(D^M h)_{\mu\nu} = 0 \rightarrow \epsilon_\mu^{\alpha\beta} \bar{\nabla}_\alpha h_{\beta\nu} + M h_{\mu\nu} = 0,\tag{22}$$

where $M = m_i$. This can be solved with the TT gauge as [23, 24]

$$h_{\mu\nu}^M = e^{-ik(\tau+\phi)-2h_L\tau} (\sinh \rho)^{-2h_L} (\tanh \rho)^{-ik} \begin{pmatrix} 1 & 0 & \frac{2}{\sinh(2\rho)} \\ 0 & 0 & 0 \\ \frac{2}{\sinh(2\rho)} & 0 & \frac{4}{\sinh^2(2\rho)} \end{pmatrix}, \quad (23)$$

where h_L is the conformal weight of graviton with mass M given by

$$h_L = \frac{M-1}{2}, \quad M \geq 1. \quad (24)$$

As was pointed out in [23], the highest weight condition of $L_1 h_{\mu\nu}^M = \bar{L}_1 h_{\mu\nu}^M = 0$ which is suited to reproduce the normalizable modes in AdS_3 spacetimes is too strong in the BTZ black hole background because the descendants of such highest weight modes have imaginary ϕ -momentum. This problem could be resolved when imposing a weaker condition so-called the chiral highest weight condition of $\bar{L}_1 h_{\mu\nu}^M = 0$ which is compatible with the TT gauge condition (15). This allows for real ϕ -momentum.

For $M = 1 (= m_1 = m_2)$, the solution is reduced to

$$h_{\mu\nu}^{M=1} = h_{\mu\nu}^L = e^{-ik(\tau+\phi)} (\tanh \rho)^{-ik} \begin{pmatrix} 1 & 0 & \frac{2}{\sinh(2\rho)} \\ 0 & 0 & 0 \\ \frac{2}{\sinh(2\rho)} & 0 & \frac{4}{\sinh^2(2\rho)} \end{pmatrix}, \quad (25)$$

which corresponds to the left-moving solution with $h_L = 0$ propagating on the BTZ black hole background [24, 25, 26]. This left-moving solution with the zero conformal weight is a cornerstone to construct the log-solution at the critical point and the \log^2 -solution at the tricritical point.

One can now construct the log-solution [5] as

$$h_{\mu\nu}^{L,\log} = \partial_M h_{\mu\nu}^M|_{M=1} = y(\tau, \rho) h_{\mu\nu}^L, \quad (26)$$

which is responsible for describing the critical GMG. Here $y(\tau, \rho)$ is defined by

$$y(\tau, \rho) = -\tau - \ln[\sinh(\rho)]. \quad (27)$$

The classical stability for this log-solution in the critical NMG has been explicitly studied in Ref. [24]. On the other hand, we need to introduce the \log^2 -solution

$$h_{\mu\nu}^{L,\log^2} = \frac{1}{2} \partial_M^2 h_{\mu\nu}^M|_{M=1} = \frac{1}{2} y^2(\tau, \rho) h_{\mu\nu}^L \quad (28)$$

for describing the tricritical GMG whose linearized equation is given by

$$\left(D^R D^L D^L D^L h^{L,\log^2} \right)_{\mu\nu} = 0. \quad (29)$$

One can easily check that

$$\begin{aligned}
(D^L h^{L, \log^2})_{\mu\nu} &= -h_{\mu\nu}^{L, \log}, \\
(D^L D^L h^{L, \log^2})_{\mu\nu} &= h_{\mu\nu}^L, \\
(D^L D^L D^L h^{L, \log^2})_{\mu\nu} &= (D^L h^L)_{\mu\nu} = 0.
\end{aligned} \tag{30}$$

One of non-trivial tasks on the BTZ black hole is to derive quasinormal modes of graviton and their frequencies. Usually, one needs to solve the second-order differential equation to find quasinormal modes with the ingoing wave at horizon and Dirichlet boundary condition at infinity. However, if one makes the second-order equation from the first-order one, sign ambiguity may appear in the mass term. Hence, it would be better to use the operator method to obtain quasinormal modes. According to the Sachs's proposal [21], the logarithmic quasinormal modes can be constructed by using the operator method as

$$h_{\mu\nu}^{L(n), \log^2}(u, v, \rho) = \left(\bar{L}_{-1} L_{-1} \right)^n h_{\mu\nu}^{L, \log^2}(u, v, \rho), \tag{31}$$

which implies that all descendants could be obtained from the chiral highest weight $h_{\mu\nu}^{L, \log^2}$ satisfying $\bar{L}_1[h_{\mu\nu}^{L, \log^2}] = 0$ by acting $\bar{L}_{-1} L_{-1}$ on it. Explicitly, the first descendant of $h_{\mu\nu}^{L, \log^2}$ is given by

$$\begin{aligned}
h_{\mu\nu}^{L(1), \log^2}(u, v, \rho) &= \left(\bar{L}_{-1} L_{-1} \right) h_{\mu\nu}^{L, \log^2}(u, v, \rho) \\
&= \frac{e^{-2\tau}}{2 \sinh^2 \rho} e^{-ik(\tau+\phi)} (\tanh \rho)^{-ik} \begin{pmatrix} f_{uu}^{L(1)} & f_{uv}^{L(1)} & \frac{f_{u\rho}^{L(1)}}{\sinh(2\rho)} \\ f_{uv}^{L(1)} & 0 & \frac{f_{v\rho}^{L(1)}}{\sinh(2\rho)} \\ \frac{f_{u\rho}^{L(1)}}{\sinh(2\rho)} & \frac{f_{v\rho}^{L(1)}}{\sinh(2\rho)} & \frac{f_{\rho\rho}^{L(1)}}{\sinh^2(2\rho)} \end{pmatrix}_{\mu\nu},
\end{aligned} \tag{32}$$

whose relevant components are given by

$$\begin{aligned}
f_{uu}^{L(1)} &= 1 + \cosh(2\rho) + [4 + 2 \cosh(2\rho) + ik(3 + \cosh(2\rho))]y(t, \rho) \\
&\quad + (2 - k^2 + 3ik)y^2(\tau, \rho), \\
f_{uv}^{L(1)} &= 2y(\tau, \rho) \left[1 + \left(1 + \frac{ik}{2} \right) y(t, \rho) \right], \\
f_{u\rho}^{L(1)} &= 2[1 + \cosh(2\rho) + 4(1 + \cosh(2\rho))y(t, \rho) + (2 + 2 \cosh(2\rho) - k^2)y^2(\tau, \rho) \\
&\quad + ik(3 + \cosh(2\rho))y(t, \rho)(1 + y(\tau, \rho))], \\
f_{v\rho}^{L(1)} &= 4y(\tau, \rho) \left[1 + \left(1 + \frac{ik}{2} \right) y(t, \rho) \right], \\
f_{\rho\rho}^{L(1)} &= 4[1 + \cosh(2\rho) + 2(2 + 3 \cosh(2\rho))y(t, \rho) + (2 + 4 \cosh(2\rho) - k^2)y^2(\tau, \rho) \\
&\quad + ik(3 + \cosh(2\rho))y(t, \rho)(1 + y(\tau, \rho))].
\end{aligned} \tag{33}$$

Next, the second descendant of $h_{\mu\nu}^{L,\log^2}(u, v, \rho)$ is derived from the operation as

$$\begin{aligned}
h_{\mu\nu}^{L(2),\log^2}(u, v, \rho) &= \left(\bar{L}_{-1}L_{-1}\right)^2 h_{\mu\nu}^{L,\log^2}(u, v, \rho) \\
&= \frac{e^{-4\tau}}{2 \sinh^4 \rho} e^{-ik(\tau+\phi)} (\tanh \rho)^{-ik} \begin{pmatrix} f_{uu}^{L(2)} & f_{uv}^{L(2)} & \frac{f_{u\rho}^{L(2)}}{\sinh(2\rho)} \\ f_{uv}^{L(2)} & f_{vv}^{L(2)} & \frac{f_{v\rho}^{L(2)}}{\sinh(2\rho)} \\ \frac{f_{u\rho}^{L(2)}}{\sinh(2\rho)} & \frac{f_{v\rho}^{L(2)}}{\sinh(2\rho)} & \frac{f_{\rho\rho}^{L(2)}}{\sinh^2(2\rho)} \end{pmatrix}_{\mu\nu},
\end{aligned} \tag{34}$$

whose components are explicitly given by

$$\begin{aligned}
f_{uu}^{L(2)} &= \frac{1}{4}[145 + 140 \cosh(2\rho) + 11 \cosh(4\rho) - k^2(27 + 20 \cosh(2\rho) + \cosh(4\rho))] \\
&+ \frac{1}{4}[274 + 200 \cosh(2\rho) + 6 \cosh(4\rho) - k^2(163 + 76 \cosh(2\rho) + \cosh(4\rho))]y(\tau, \rho), \\
&+ (24 + 12 \cosh(2\rho) - k^2(30 + 7 \cosh(2\rho)) + k^4)y^2(\tau, \rho) \\
&+ \frac{ik}{4}(125 + 108 \cosh(2\rho) + 7 \cosh(4\rho)) \\
&+ \frac{ik}{4}[367 + 220 \cosh(2\rho) + 5 \cosh(4\rho) - 8k^2(3 + \cosh(2\rho))]y(\tau, \rho) \\
&+ ik(44 + 16 \cosh(2\rho) - k^2(9 + \cosh(2\rho)))y^2(\tau, \rho),
\end{aligned} \tag{35}$$

$$\begin{aligned}
f_{uv}^{L(2)} &= 18 + 14 \cosh(2\rho) + 2[26 + 16 \cosh(2\rho) - k^2(5 + \cosh(2\rho))]y(t, \rho) \\
&+ 2(12 + 6 \cosh(2\rho) - k^2(7 + \cosh(2\rho)))y^2(\tau, \rho) \\
&+ 4ik(2 + \cosh(2\rho)) + 2ik(23 + 9 \cosh(2\rho))y(t, \rho) \\
&+ 2ik(16 + 5 \cosh(2\rho) - k^2)y^2(\tau, \rho),
\end{aligned} \tag{36}$$

$$\begin{aligned}
f_{u\rho}^{L(2)} &= \frac{1}{2}[177 + 212 \cosh(2\rho) + 35 \cosh(4\rho) - k^2(27 + 20 \cosh(2\rho) + \cosh(4\rho))] \\
&+ \frac{1}{2}[358 + 408 \cosh(2\rho) + 50 \cosh(4\rho) - k^2(167 + 116 \cosh(2\rho) + 5 \cosh(4\rho))]y(\tau, \rho) \\
&+ [66 + 72 \cosh(2\rho) + 6 \cosh(4\rho) - k^2(63 + 42 \cosh(2\rho) + \cosh(4\rho)) + 2k^4]y^2(\tau, \rho) \\
&+ \frac{ik}{2}(133 + 140 \cosh(2\rho) + 15 \cosh(4\rho)) \\
&+ \frac{ik}{2}[411 + 404 \cosh(2\rho) + 33 \cosh(4\rho) - 8k^2(3 + \cosh(2\rho))]y(\tau, \rho) \\
&+ ik[103 + 96 \cosh(2\rho) + 5 \cosh(4\rho) - 6k^2(3 + \cosh(2\rho))]y^2(\tau, \rho),
\end{aligned} \tag{37}$$

$$f_{vv}^{L(2)} = 2[2 + 10y(\tau, \rho) + (6 - k^2)y^2(\tau, \rho) + 4iky(\tau, \rho) - 5iky^2(\tau, \rho)], \tag{38}$$

$$\begin{aligned}
f_{v\rho}^{L(2)} &= 4\{9(1 + \cosh(2\rho)) + [26 + 26 \cosh(2\rho) - k^2(5 + \cosh(2\rho))]\}y(t, \rho) \\
&+ (12 + 12 \cosh(2\rho) - k^2(7 + 2 \cosh(2\rho)))y^2(\tau, \rho) \\
&+ 2ik(2 + \cosh(2\rho)) + ik(23 + 13 \cosh(2\rho))y(t, \rho) \\
&+ ik(16 + 10 \cosh(2\rho) - k^2)y^2(\tau, \rho)\}, \tag{39}
\end{aligned}$$

$$\begin{aligned}
f_{\rho\rho}^{L(2)} &= 217 + 284 \cosh(2\rho) + 67 \cosh(4\rho) - k^2(27 + 20 \cosh(2\rho) + \cosh(4\rho)) \\
&+ [482 + 616 \cosh(2\rho) + 134 \cosh(4\rho) - 3k^2(57 + 52 \cosh(2\rho) + 3 \cosh(4\rho))]\}y(\tau, \rho) \\
&+ 4[48 + 60 \cosh(2\rho) + 12 \cosh(4\rho) - k^2(34 + 35 \cosh(2\rho) + 2 \cosh(4\rho)) - k^4]y^2(\tau, \rho) \\
&+ ik(141 + 172 \cosh(2\rho) + 23 \cosh(4\rho)) \\
&+ ik[471 + 588 \cosh(2\rho) + 77 \cosh(4\rho) - 8k^2(3 + \cosh(2\rho))]\}y(\tau, \rho) \\
&+ 4ik[64 + 80 \cosh(2\rho) + 10 \cosh(4\rho) - k^2(9 + 5 \cosh(2\rho))]\}y^2(\tau, \rho). \tag{40}
\end{aligned}$$

From these expressions, one can deduce the n^{th} -order descendant as

$$\begin{aligned}
h_{\mu\nu}^{L(n), \log^2}(u, v, \rho) &= \left(\bar{L}_{-1}L_{-1}\right)^n h_{\mu\nu}^{L, \log^2}(u, v, \rho) \\
&= \frac{e^{-2n\tau}}{2 \sinh^{2n}\rho} e^{-ik(\tau+\phi)} (\tanh\rho)^{-ik} F_{\mu\nu}^{L(n)}(\rho), \tag{41}
\end{aligned}$$

where $F_{\mu\nu}^{L(n)}(\rho)$ is the corresponding n^{th} - order matrix. As a result, we read off the log-square quasinormal frequencies of the graviton at the tricritical point 1 as

$$\omega_L^n = k - i4\pi T_L n, \quad n \in Z, \tag{42}$$

which is the same expression for the spin-2 graviton $h_{\mu\nu}$ at the critical point [21].

At this stage, it is appropriate to comment on the right-moving solution and log-square quasinormal modes at the tricritical point 2. The right-moving solution and its corresponding log-square solution can be easily constructed by the substitution of $u \rightarrow v$, $L \rightarrow R$ ($\phi \rightarrow -\phi$, $k \rightarrow -k$) in Eqs. (23) and (25). Actually, one starts with

$$h_{\mu\nu}^M = e^{ik(\tau+\phi)-2h_R\tau} (\sinh\rho)^{-2h_R} (\tanh\rho)^{ik} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \frac{2}{\sinh(2\rho)} \\ 0 & \frac{2}{\sinh(2\rho)} & \frac{4}{\sinh^2(2\rho)} \end{pmatrix}, \tag{43}$$

where h_R is the conformal weight of graviton with mass M given by

$$h_R = -\frac{M+1}{2}, \quad M \leq -1. \tag{44}$$

For $M = -1$, $h_{\mu\nu}^M$ leads to $h_{\mu\nu}^R$

$$h_{\mu\nu}^R = e^{ik(\tau+\phi)} (\tanh \rho)^{ik} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \frac{2}{\sinh(2\rho)} \\ 0 & \frac{2}{\sinh(2\rho)} & \frac{4}{\sinh^2(2\rho)} \end{pmatrix}. \quad (45)$$

With the log and log-square solutions of

$$h_{\mu\nu}^{R,\text{Log}} = -y(\tau, \rho) h_{\mu\nu}^R, \quad (46)$$

$$h_{\mu\nu}^{R,\text{Log}^2} = \frac{1}{2} y^2(\tau, \rho) h_{\mu\nu}^R, \quad (47)$$

the tricritical GMG at the point 2 is described by

$$\left(D^L D^R D^R D^R h^{R,\log^2} \right)_{\mu\nu} = 0. \quad (48)$$

One can also easily check that

$$\begin{aligned} (D^R h^{R,\log^2})_{\mu\nu} &= -h_{\mu\nu}^{R,\log}, \\ (D^R D^R h^{R,\log^2})_{\mu\nu} &= h_{\mu\nu}^R, \\ (D^R D^R D^R h^{R,\log^2})_{\mu\nu} &= (D^R h^R)_{\mu\nu} = 0. \end{aligned} \quad (49)$$

The succeeding descendants of the log-square quasinormal modes at the tricritical point 2 is simply derived by applying the mentioned substitution, and finally yield the quasinormal frequencies as

$$\omega_R^n = -k - i4\pi T_R n, \quad n \in \mathbb{Z}. \quad (50)$$

4 Log-square boundary conditions

First of all, we review the \log^2 -boundary condition on the AdS_3 spacetimes. On the AdS_3 background, the \log^2 -solution does not obey either the Brown-Henneaux boundary or the log-boundary conditions. Hence, Liu and Sun [15] have introduced the \log^2 -boundary condition,

$$\tilde{h}_{\mu\nu}^{\log^2} = \begin{pmatrix} \rho^2 & 1 & \rho^2 e^{-2\rho} \\ 1 & 1 & e^{-2\rho} \\ \rho^2 e^{-2\rho} & e^{-2\rho} & e^{-2\rho} \end{pmatrix}_{\mu\nu}, \quad (51)$$

which is a relaxed form obtained by replacing $\tilde{h}_{uu} = 1$ and $\tilde{h}_{u\rho} = e^{-2\rho}$ in the Brown-Henneaux boundary by $\tilde{h}_{uu}^{\log^2} = \rho^2$ and $\tilde{h}_{u\rho}^{\log^2} = \rho^2 e^{-2\rho}$ on the AdS_3 spacetimes. It is worth noting that the \log^2 -boundary condition is different from the Brown-Henneaux boundary in the CFT.

Similarly, we conjecture that for \log^2 -quasinormal modes, its asymptotic boundary condition should differ from those of ordinary quasinormal modes. In the case of quasinormal modes around the BTZ black hole, one expects that all quasinormal modes fall off exponentially in time τ and for large radial distance ρ , together with ingoing modes at the horizon. The asymptotic behavior for the highest weight mode (25) for constructing the ordinary quasinormal modes is given by [23]

$$h_{\mu\nu}^L \sim \begin{pmatrix} 1 & 0 & e^{-2\rho} \\ 0 & 0 & 0 \\ e^{-2\rho} & 0 & e^{-4\rho} \end{pmatrix}_{\mu\nu}. \quad (52)$$

On the other hand, the asymptotic behavior for the highest weight mode (28) for constructing \log^2 -quasinormal modes takes the form

$$h_{\mu\nu}^{L,\log^2} \sim \rho^2 \begin{pmatrix} 1 & 0 & e^{-2\rho} \\ 0 & 0 & 0 \\ e^{-2\rho} & 0 & e^{-4\rho} \end{pmatrix}_{\mu\nu}. \quad (53)$$

Here we note that (53) involves h_{uu}^{L,\log^2} which is quadratically growing for large ρ . This might be tempted to disqualify as a quasinormal mode because all quasinormal modes fall off for large radial distance ρ . Considering Eq. (51), $h_{uu}^{L,\log^2} \sim \rho^2$ in Eq. (53) shows a similar behavior as in $\tilde{h}_{uu}^{\log^2} \sim \rho^2$ on the AdS_3 spacetimes. However, from the observation of the first descendent (32) of the \log^2 solution, its asymptotic form is given by

$$h_{\mu\nu}^{L(1),\log^2} \sim \rho^2 \begin{pmatrix} -\frac{1}{\rho} & e^{-2\rho} & e^{-2\rho} \\ e^{-2\rho} & 0 & e^{-4\rho} \\ e^{-2\rho} & e^{-4\rho} & e^{-4\rho} \end{pmatrix}_{\mu\nu}, \quad (54)$$

which contains still $h_{uu}^{L(1),\log^2} \sim \rho$ -term which gives rise to divergences at infinity. In order to see how this divergence is tamed, we obtain asymptotic form of the second descendent (34) of the \log^2 -solution

$$h_{\mu\nu}^{L(2),\log^2} \sim \rho^2 \begin{pmatrix} -\frac{1}{\rho} & e^{-2\rho} & e^{-2\rho} \\ e^{-2\rho} & e^{-4\rho} & e^{-4\rho} \\ e^{-2\rho} & e^{-4\rho} & e^{-4\rho} \end{pmatrix}_{\mu\nu} \quad (55)$$

where ρ still appears in the (uu) -element. Successively, it is appropriate to compute the third descendants of $h_{\mu\nu}^{L(3),\log^2}$. Its asymptotic behavior is exactly the same with the asymptotic form (55) of the second descendent. As a result, similar to the previous work [26], we expect

that all higher order descendants with $n > 3$ behave as the second descendent shows. This implies that one could not eliminate the linear divergence in $h_{uu}^{L(2), \log^2}$ if one considers the tricritical gravity, as in the \log^2 -boundary condition on the AdS_3 . However, this may give rise to some difficulty in identifying the corresponding dual operator in the CFT [21].

5 Rank-3 LCFT

First of all, the log gravity at the tricritical point could be dual to a rank-3 LCFT with $c_L = 0$ on the boundary [18, 10]. The rank-3 LCFT [27, 28] is composed of three operators $\{\mathcal{O}^L(z), \mathcal{O}^{\log}(z), \mathcal{O}^{\log^2}(z)\}$ which are denoted as $\{C(z), D(z), E(z)\}$, for simplicity. The two-point functions of these operators take the forms

$$\langle C(z)C(0) \rangle = \langle C(z)D(0) \rangle = 0, \quad (56)$$

$$\langle C(z)E(0) \rangle = \langle D(z)D(0) \rangle = \frac{a_L}{2z^{2h_L}}, \quad (57)$$

$$\langle D(z)E(0) \rangle = -\frac{a_L \log z}{z^{2h_L}}, \quad (58)$$

$$\langle E(z)E(0) \rangle = \frac{a_L \log^2 z}{z^{2h_L}}, \quad (59)$$

which form a rank-3 Jordan cell. Schematically, these two-point correlation functions are represented by

$$\langle \mathcal{O}^i \mathcal{O}^j \rangle \sim \begin{pmatrix} 0 & 0 & \text{CFT} \\ 0 & \text{CFT} & L \\ \text{CFT} & L & L^2 \end{pmatrix}, \quad (60)$$

where $i, j = L, \log, \log^2$, CFT denotes the CFT two-point function (57), L represents (58), and L^2 denotes (59).

At this stage, we stress that (56)-(59) show a rank-3 zero temperature LCFT. Even though one knows a rank-3 zero temperature LCFT, it is a non-trivial task to construct a rank-3 finite temperature LCFT whose zero temperature limits correspond to (56)-(59). If one knows the latter, one could read off the \log^2 quasinormal frequencies from the finite temperature LCFT exactly. We note that a rank-2 finite temperature LCFT was known in [20].

In this section, we derive the quasinormal frequencies $\omega_L^n = k - i4\pi T_L n$ (42) of the graviton using the finite temperature CFT. Here we wish to use the finite temperature CFT but not the finite temperature LCFT. In order to derive quasinormal modes, we focus at the location of the poles in the momentum space for the retarded two-point functions $G_R^{CE}(\tau, \sigma)$, $G_R^{DD}(\tau, \sigma)$, $G_R^{DE}(\tau, \sigma)$, and $G_R^{EE}(\tau, \sigma)$ [21]. It is important to recognize that as is shown in Eq. (57), $G_R^{CE}(\tau, \sigma)$ and $G_R^{DD}(\tau, \sigma)$ are identical with that of the two-point function in the

finite temperature CFT [19]. The momentum space representation can be read off from the commutator whose pole structure is given by

$$\mathcal{D}^{CE}(p_+) \sim \mathcal{D}^{DD}(p_+) \propto \Gamma\left(h_L + i\frac{p_+}{2\pi T_L}\right) \Gamma\left(h_L - i\frac{p_+}{2\pi T_L}\right), \quad (61)$$

where $p_+ = (\omega - k)/2$ and $T_L = 1/2\pi$ for the BTZ black hole. This function has poles in both the upper and lower half of the ω -plane. It turned out that the poles located in the lower half-plane are the same as those of the retarded two-point functions $G_R^{CE}(\tau, \sigma)$ and $G_R^{DD}(\tau, \sigma)$. Restricting the poles in Eq. (61) to the lower half-plane, we find one set of simple poles

$$\omega_s = k - i4\pi T_L(n + h_L), \quad (62)$$

with $n \in \mathbb{N}$. This set of poles characterizes the decay of the perturbation on the CFT side [19], while (62) was first derived from the scalar perturbation around the BTZ black hole [29] and scalar wave-falloff was discussed in AdS_3 spacetimes [30]. At this stage, we would like to mention that the same quasinormal frequencies (62) was obtained by a slightly different operator method based on the hidden conformal symmetry appeared in the linearized equation [31, 32], which is not an underlying symmetry of the spacetime itself as shown in the line element (12). Sach and Solodukhin [23] have used the latter symmetry to construct quasinormal modes and to find quasinormal frequencies, followed by us. Even though the symmetry group is the same as $\text{SL}(2, R)$, their origin is different.

Furthermore, $G_R^{DE}(t, \sigma)$ can be inferred by noting [21]

$$\langle D(x)E(0) \rangle = \frac{\partial}{\partial h_L} \langle C(x)E(0) \rangle. \quad (63)$$

Then, this implies that its momentum space representation takes the form

$$\mathcal{D}^{DE}(p_+) \propto \Gamma'(h_L + ip_+) \Gamma(h_L - ip_+) + \Gamma(h_L + ip_+) \Gamma'(h_L - ip_+), \quad (64)$$

where the prime ($'$) denotes the differentiation with respect to h_L . We mention that (64) is a relevant part for extracting pole structure, but its explicit form appeared in [20]. The poles in the lower half-plane are relevant to deriving quasinormal modes. We note that $\mathcal{D}^{DE}(p_+)$ has double poles, while $\mathcal{D}^{CE}(p_+)$ has simple poles at the same location. These double poles are responsible for explaining the linear-time dependence in $y(t, \rho)$ of the corresponding quasinormal modes (26). Restricting the double poles in Eq. (64) to the lower half-plane, we find one set of double poles

$$\omega_d = k - i4\pi T_L(n + h_L). \quad (65)$$

Finally, $G_R^{EE}(t, \sigma)$ can be inferred by noting

$$\langle E(x)E(0) \rangle = \frac{1}{2} \frac{\partial^2}{\partial h_L^2} \langle C(x)E(0) \rangle, \quad (66)$$

which implies that its momentum space representation takes the form

$$\begin{aligned} \mathcal{D}^{EE}(p_+) &\propto \Gamma''(h_L + ip_+) \Gamma(h_L - ip_+) + 2\Gamma'(h_L + ip_+) \Gamma'(h_L - ip_+) \\ &+ \Gamma(h_L + ip_+) \Gamma''(h_L - ip_+). \end{aligned} \quad (67)$$

The poles in the lower half-plane are relevant to deriving quasinormal modes. We mention that $\mathcal{D}^{EE}(p_+)$ has triple poles, while $\mathcal{D}^{CE}(p_+)$ has simple poles at the same location. These triple poles are responsible for the quadratic-time dependence in $y^2(t, \rho)$ of the corresponding quasinormal modes (28). Restricting the triple poles in Eq. (67) to the lower half-plane, we find one set of triple poles

$$\omega_t = k - i4\pi T_L(n + h_L). \quad (68)$$

All these quasinormal frequencies ($\omega_s, \omega_d, \omega_t$) are for the scalar with $h_L = 2$. For the tensor perturbation, its conformal weight is given by $h_L = (M - 1)/2$ as (24) is shown. At the tricritical point $M = 1$, one has $h_L = 0$ and finally, plugging it to (68) leads to (42).

6 Truncation of the tricritical GMG

On the AdS_3 background, in order to remove the non-unitary LCFT, one restricts the theory to the zero of Abbott-Deser-Tekin charge ($Q_L = 0$). This corresponds to truncating tricritical GMG, which amounts to reducing the \log^2 -boundary conditions to log-boundary condition. After the truncation, the two-point correlation functions take the form

$$\langle \mathcal{O}^i \mathcal{O}^j \rangle \sim \begin{pmatrix} 0 & 0 \\ 0 & \text{CFT} \end{pmatrix}, \quad (69)$$

which implies that the left-moving sector involves a non-trivial two-point correlator

$$\langle D(z)D(0) \rangle \equiv \langle \mathcal{O}^{\log}(z) \mathcal{O}^{\log}(0) \rangle = \frac{a_L}{2z^{2h_L}}. \quad (70)$$

This is unitary and thus, the non-unitary issue is resolved by truncating the tricritical point of the GMG.

What happens for the quasinormal modes when truncating the \log^2 - quasinormal modes on the BTZ black hole? At this stage, it is not easy to answer to this question because we did

not construct a rank-3 finite temperature LCFT whose zero temperature limits correspond to (56)-(59). If this finite temperature LCFT is constructed, one may apply the above truncation process to obtain the quasinormal modes (62) which is obtained from simple poles existing in the finite temperature CFT. This may correspond to CFT in (69).

7 Discussions

As was mentioned in the introduction, there is a clear difference in the AdS/LCFT correspondence between “tricritical gravity on the $\text{AdS}_3 \rightarrow$ a rank-3 LCFT” and “tricritical gravity on the BTZ black hole \rightarrow a rank-3 finite temperature LCFT”. We investigated the tricritical GMG by following the latter line because the former was almost confirmed.

We have obtained log-square quasinormal modes and frequencies of a graviton around the BTZ black hole at the tricritical GMG by employing the operator method. The log-square quasinormal frequencies are also obtained by using a finite temperature CFT. This shows the AdS/LCFT correspondence at the tricritical point approximately.

Even though one knows a rank-3 zero temperature LCFT (56)-(59), it is a non-trivial task to construct a rank-3 finite temperature LCFT whose zero temperature limits correspond to (56)-(59). If one knows the latter, one could read off the \log^2 -quasinormal frequencies from the finite temperature LCFT precisely. In this case, one could apply the truncation process to obtain the quasinormal modes (62) which is obtained from simple poles existing in the finite temperature CFT, together with removing (65) and (68).

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